M75

Burger

Derivatives of Trigonometric Functions

$$\boxed{\frac{d}{dx}(\sin x)} \rightarrow \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

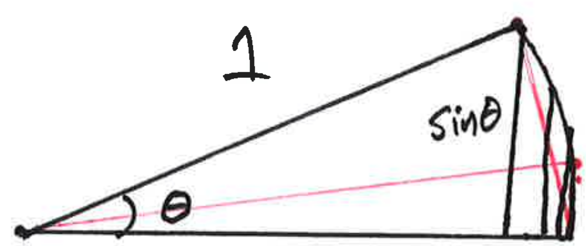
Recall $\sin(A+B) = \sin A \cos B + \sin B \cos A$

$$= \lim_{h\to 0} \frac{\sin x \cosh + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \boxed{\frac{\sin x(\cosh - 1)}{h}} + \lim_{h\to 0} \boxed{\frac{\sin h}{h}} \cos x$$

Important limits!

$$\lim_{h\to 0} \frac{\sin h}{h}$$



$\sin\theta = \frac{opp}{1}$

$= opp.$

Do you know what radians are?

$S = r \cdot \theta$

Arc length of circle

As $\theta \to 0$ Bottom Distance $\to 1$

$\sin\theta \to \theta$ Arclength.

$\frac{\sin\theta}{\theta} \to 1$

$C = 2\pi r$, when $r = 1$ $C = 2\pi = \theta$ radians

Once you are convinced

$\frac{\sin x}{x} \to 1$ as $x \to 0$.

Then: $\lim_{x \to 0} \frac{(\cos x - 1)(\cos x + 1)}{x \quad (\cos x + 1)} =$

$\lim_{x \to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)} = \lim_{x \to 0} \frac{-\sin^2 x}{x(\cos x + 1)}$

$= \lim_{x \to 0} \left[-\frac{\sin x}{x} \right] \cdot \left[\frac{\sin x}{\cos x + 1} \right]$

$= -1 \cdot \left[\frac{0}{2} \right] = -1 \cdot 0 = 0$ ✓

$\therefore \lim_{h \to 0} \sin x \underbrace{\frac{(\cos h - 1)}{h}}_{0} + \lim_{h \to 0} \underbrace{\left[\frac{\sin h}{h} \right]}_{1} \cos x$

$\therefore$

$\frac{D(\sin x)}{dx} = \cos x$ ∎

$\frac{D(\cos x)}{dx} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$

2.

$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h \to 0} \frac{\sin h \sin x}{h}$$

$$= \cos x \cdot 0 - 1 \cdot \sin x = -\sin x$$

$$\frac{D(\cos x)}{dx} = -\sin x.$$

With these building Blocks And the Quotient rule we can derive the others.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \sin x - \cos x (\sin x)'}{\sin^2 x}$$

$$= \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

Similarly $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

$$\frac{d}{dx}\left((\cos x)^{-1}\right)$$

Use chain rule! $\quad -(\cos x)^{-2} \cdot (-\sin x)$

$$= \frac{(-1)(-1)\sin x}{\cos x \cos x}$$

$$= \tan x \cdot \frac{1}{\cos x} = \tan x \cdot \sec x$$

or $\sec x \cdot \tan x$.

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left((\sin x)^{-1}\right)$$

$$= -(\sin x)^{-2} \cdot \cos x$$

$$= \frac{-\cos x}{\sin x \cdot \sin x} = -\cot x \cdot \csc x$$

or $-\csc x \cdot \cot x$

Higher order Derivatives.

Find $\dfrac{d^{(40)}y}{dx^{(40)}}$ and $\dfrac{d^{(42)}y}{dx^{42}}$

when $y = \sin x$.

<u>soln.</u> $\dfrac{dy^{(1)}}{dx} = \cos x$ $\quad \dfrac{d^2y}{dx^2} =$ $(\cos x)' = -\sin x$

3rd Derivative:

$$\frac{d^3y}{dx^3} = \frac{d(-\sin x)}{dx} = -1 \cdot \cos x = -\cos x$$

$$\frac{d^4y}{dx^4} = \frac{d(-\cos x)}{dx} = -1 \cdot \frac{d\cos x}{dx} = (-1)(-\sin x) = \sin x.$$

So every 4 th one is Back to $\sin x$.

$$\frac{d^5y}{dx^5} = \frac{d^2y}{dx^2} = \cos x.$$

So $\dfrac{d^{40}y}{dx^{40}} = \sin x$ And $\dfrac{d^{42}y}{dx^{42}} = -\sin x.$

$\frac{42}{4} = 10$ r 2

5.

Tricks for $\dfrac{\sin kx}{x}$ limits.

Example 1 in e-book.

a. $\lim\limits_{x\to 0} \dfrac{\sin 4x}{x}$ $\Longrightarrow$ we know

$\lim\limits_{x\to 0} \dfrac{\sin 4x}{4x} = 1$ $\qquad \dfrac{\sin *}{*} \to 1$ as. $* \to 0$

so make it look like $\dfrac{\sin 4x}{x}$

$$4 \cdot \lim_{x\to 0} \frac{\sin 4x}{4x} = 4 \cdot 1 = 4$$

$$= \lim_{x\to 0} \frac{\cancel{4} \cdot \sin 4x}{\cancel{4}x}$$

$$= \lim_{x\to 0} \frac{\sin 4x}{x}.$$

b. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 5x} \cdot \dfrac{5x}{5x} \cdot \dfrac{3x}{3x} = \dfrac{5x}{\sin 5x} \cdot \dfrac{\sin 3x}{3x} \cdot \dfrac{3\cancel{x}}{5\cancel{x}}$

$\Longrightarrow 1 \cdot 1 \cdot 3/5 = 3/5$ ✓

Q. Check. $\lim\limits_{x\to 0} \dfrac{\tan 2x}{x} = \lim\limits_{x\to 0} \dfrac{\sin 2x}{\cos 2x \cdot x} = 2 \cdot \dfrac{1}{1} = 2$ ✓

2

Select.

Homework Problems (suggestions) 3-5

#7 $y = e^{8x} \sin x$

Find y''.

$$y' = (e^{8x})' \sin x + e^{8x} (\sin x)'$$

$$= 8e^{8x} \cdot \sin x + e^{8x} \cdot \cos x$$

you need to take derivative again!

#3. $y = 16 \cos x \sin x$

$$y' = 16 \left[(\cos x)' \cdot \sin x + \cos x (\sin x)' \right]$$

etc. $= 16\left(-\sin^2 x + \cos^2 x\right)$

#4. $y = \dfrac{x \sin x}{3 + \cos x}$

Quotient AND product rule!

Polar coordinates!

5.

$$r = \cos\theta \cot\theta$$

$$\frac{dr}{d\theta} = (\cos\theta)' \cot\theta + \cos\theta (\cot\theta)'$$

$$= -\sin\theta \cot\theta + \cos\theta(-\csc^2\theta)$$

#6

$$y = \frac{8\tan t}{9 + 9\sec t}$$

Quotient rule!